JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 29, No. 1, January–February 2006

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Statistical Distribution of Keplerian Velocities

Dario Izzo*
ESA, 2201 AZ Noordwijk, The Netherlands

Introduction

THE use of generalized functions, in particular of Dirac's delta functions, has already been proposed[1–5] as a useful tool in simplifying the derivations of probability distribution functions related to the statistical description of Keplerian orbits. The formalism that derives from an extensive use of these functions permits an easy conversion from classical astrodynamic formulas to probability distributions. The methodology was independently discovered in different forms by Au and Tam,[6] Meshcheryakov,[4] and Izzo,[1] and it has been used in various applications[1–3,5] mainly related to the description of collision probabilities and to the modeling of the space debris environment. The limitations in the description of the Molnyia and geostationary orbits of the orbital debris environment model proposed by Kessler et al.[7] were alleviated by using results derived with this methodology.[3] These results also generalized previous expressions[8–10] commonly used in space flight mechanics problems dealing with the statistics of a large number of orbits and objects. In the evaluation of collision risks as well as in the problem of inferring the orbital motion of space debris from a measurement campaign such as the one carried by the Long Duration Exposure Facility or by the more recent Space Dust[11] instrument, it is often necessary to have statistical models also on the velocities of orbiting objects. In this Note Dirac's delta formalism is applied to derive the absolute velocity distribution of an orbiting object having uncertain orbital parameters.

Methodology

Using the change of variable technique on random variables[12] to evaluate the probabilistic distribution of quantities that are related to them is common. Au and Tam[6] note that this technique has a limitation in that it naturally deals only with a number of new variables equal to the number of old variables and with one-to-one transformations. In these cases, the technique becomes less straightforward, and derivations may get quite lengthy and complicated. One solution could be using a fundamental property of Dirac's delta functions, and in particular, the following theorem:

$$\delta[x - \hat{x}(y)] = \sum_i \frac{\delta[y - y_i]}{|\mathrm{d}\hat{x}/\mathrm{d}y|_{y=y_i}} \tag{1}$$

where y_i are all of the solutions in y of the equation $x = \hat{x}(y)$. The theorem has also a multivariate form,

$$\delta[\boldsymbol{x} - \hat{\boldsymbol{x}}(\boldsymbol{y})] = \sum_i \frac{\delta[\boldsymbol{y} - \boldsymbol{y}_i]}{|J|_{y=y_i}} \tag{2}$$

where $J = \partial\hat{x}_i/\partial y_j$ is the Jacobian of the transformation $\hat{\boldsymbol{x}}(\boldsymbol{y})$ and $\boldsymbol{y}_i$ are the solutions to the equation $\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{y})$.

Dirac's delta functions are used to define the probability density function associated with a deterministic process. The desired random variables are then disintegrated from the process and the integral is turned into a summation by Eq. (1). Note that the term disintegration refers to a standard procedure in the theory of probability that is more commonly called randomization. We may define this procedure with the following equation:

$$\rho(x) = \int_{-\infty}^{\infty} \rho(x|y) f(y)\,\mathrm{d}y$$

where $\rho(x)$ is the probability density function of the random variable x, $\rho(x|y)$ is the probability density function of x conditioned to the knowledge of y, and $f(y)$ is the distribution of y.

Consider a simple linear system,

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$

with initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}_0$. Its solution may be defined by introducing the matrix exponential, as $\hat{\boldsymbol{x}}(\boldsymbol{x}_0, t) = e^{At}\boldsymbol{x}_0$. The probability density function associated with the state $\boldsymbol{x}$ is

$$\rho(\boldsymbol{x}|\boldsymbol{x}_0, t) = \delta[\boldsymbol{x} - \hat{\boldsymbol{x}}(\boldsymbol{x}_0, t)]$$

We may now think of the initial condition $\boldsymbol{x}_0$ as a random variable with probability distribution function $f(\boldsymbol{x}_0)$, and we may, therefore, disintegrate it by writing

$$\rho(\boldsymbol{x}|t) = \int_{x_0 \in D} \delta[\boldsymbol{x} - \hat{\boldsymbol{x}}(\boldsymbol{x}_0, t)] f(\boldsymbol{x}_0)\,\mathrm{d}\boldsymbol{x}_0$$

where D is the domain of f. When Eq. (2) is used, it is possible to write

$$\rho(\boldsymbol{x}|t) = \int_{x_0 \in D} \delta\left[\boldsymbol{x}_0 - \boldsymbol{x}_0^*\right] f\left(\boldsymbol{x}_0^*\right) \frac{1}{|e^{At}|}\,\mathrm{d}\boldsymbol{x}_0$$

where $\boldsymbol{x}_0^*$ is the only solution of equation $\boldsymbol{x} = \hat{\boldsymbol{x}}(\boldsymbol{x}_0, t)$ given by $\boldsymbol{x}_0^* = e^{At^{-1}}\boldsymbol{x}$. Thus, we write a final expression for the probability density function associated with the variable $\boldsymbol{x}$ whenever the initial conditions of the dynamic system that describe the time evolution of $\boldsymbol{x}$ are considered random and have to be disintegrated from the process,

$$\rho(\boldsymbol{x}|t) = f\left(e^{At^{-1}}\right)\boldsymbol{x} \Big/ |e^{At}|$$

Note that in this case the same expression could be reached using a standard change of variable technique. The transformation $\boldsymbol{x} = e^{At}\boldsymbol{x}_0$ from $\boldsymbol{x}_0$ to $\boldsymbol{x}$ is in fact a one-to-one transformation and the dimension of the vector $\boldsymbol{x}$ is the same as that of the vector $\boldsymbol{x}_0$. (The numbers of old and new variables are the same.) If we wish to consider time as random, even the simple generic linear system considered does

Received 2 May 2005; revision received 18 July 2005; accepted for publication 19 July 2005.

*Research Fellow, Advanced Concepts Team, European Space Research and Technology Centre, Keplerlaan 1.

not admit a one-to-one transformation and Dirac deltas may provide some useful help in the treatment of these situations by reducing the analytical computations.

Applications to Space Flight Mechanics

The methodology described in the preceding section has been applied to different problems related to orbital mechanics in general and to the description of the space debris environment.[1–5] The spatial density of a large family of orbiting objects has been derived analytically for geostationary, low Earth orbit (LEO), and Molnyia satellites. Additional results are derived here to show the use of Dirac's delta when disintegrating random variables from a given stochastic process and, in particular, in dealing with uncertainties related to the orbital parameters and their effects on velocity distributions.

Hodograph Plane

Let us briefly recall standard results on the velocity vector along a Keplerian orbit. In particular, we start from the definitions of two important constants,

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v}, \qquad \mu \boldsymbol{e} = \boldsymbol{v} \times \boldsymbol{h} - \mu(\boldsymbol{r}/r) \tag{3}$$

that is, the angular momentum vector and the Laplace vector. Taking the vector product between these two quantities, we get

$$(h/\mu)\boldsymbol{v} = e \sin \nu \boldsymbol{i}_\rho + (1 + e \cos \nu)\boldsymbol{i}_\theta$$

where ν is the true anomaly along the orbit and $\boldsymbol{i}_\rho$ and $\boldsymbol{i}_\theta$ are the unit vectors of the local horizontal local vertical frame. The preceding expression describes the relation between the velocity and the true anomaly along a Keplerian orbit. For orbits it is common to introduce the argument of latitude $\theta = \nu + \omega$.

Velocity Magnitude Distribution

Consider a satellite on a Keplerian orbit and the probability density function associated with the variable $k = v^2$. Following the methodology outlined earlier, first a deterministic process is considered:

$$\rho(k|\boldsymbol{\alpha}, t) = \delta_k[k - \hat{k}(\boldsymbol{\alpha}, t)]$$

where $\boldsymbol{\alpha}$ is a vector containing the initial conditions defining the Keplerian orbit (where any set of orbital parameters might be considered) and $\hat{k}$ is the squared velocity along a Keplerian orbit. From the energy equation $v^2/2 - \mu/r = -\mu/2a$ and taking into account that $r = p/\{1 + e\cos[\theta(t) - \omega]\}$, we get

$$\hat{k}(\boldsymbol{\alpha}, t) = \mu\big((2/p)\{1 + e\cos[\theta(t) - \omega]\} - (1/a)\big) \tag{4}$$

Then the time t is considered as random and uniformly distributed over one orbital period so that the variable is disintegrated from the process,

$$\rho(k|\boldsymbol{\alpha}) = \frac{1}{T}\int_0^T \delta_k[k - \hat{k}(\boldsymbol{\alpha}, t)]\,\mathrm{d}t$$

To evaluate the preceding integral, we change the variable in the Dirac delta function, which is done by applying Eq. (1),

$$\rho(k|\boldsymbol{\alpha}) = \frac{1}{T}\int_0^T \sum_i \frac{\delta_t[t - t_i]}{|\mathrm{d}\hat{k}/\mathrm{d}t|_{t=t_i}}\,\mathrm{d}t \tag{5}$$

where we must sum all of the solutions in t to the equation $\hat{k}(\boldsymbol{\alpha}, t) = k$. We use Eq. (4) and the following identities:

$$e = (v_p - v_a)/(v_p + v_a), \qquad a = \mu/v_a v_p$$

where v_a and v_p are the apogee and perigee velocities, to derive a simple expression for the desired solutions,

$$\cos[\theta(t_i) - \omega] = \frac{2k - v_a^2 - v_p^2}{v_p^2 - v_a^2} \tag{6}$$

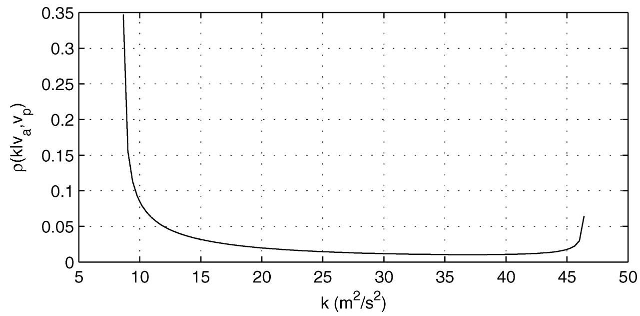

Fig. 1 Probability density function associated with the variable $k = v^2$ on an Earth orbit: $e = 0.4$ and $a = 20{,}000$ km.

Two of the solutions expressed implicitly by Eq. (6) will be contained within the time-integration interval of one period, giving nonzero terms in Eq. (5). We also have

$$\sin[\theta(t_i) - \omega] = \pm\big[2\big/\big(v_p^2 - v_a^2\big)\big]\sqrt{\big(k - v_a^2\big)\big(v_p^2 - k\big)}$$

$$1/r = (1/2\mu)(k + v_a v_p)$$

so that we may evaluate the derivatives

$$\left|\frac{\mathrm{d}\hat{k}}{\mathrm{d}t}\right|_{t=t_i} = \frac{2\mu}{p} e \sin[\theta(t_i) - \omega]\dot{\theta}(t_i)$$

$$= \frac{2\mu}{p} e \sin[\theta(t_i) - \omega] b \sqrt{\frac{\mu}{a}} \frac{1}{r^2}$$

$$= \frac{\big(k + v_a^2 v_p^2\big)^2}{2\mu(v_a + v_p)} \sqrt{\big(k - v_a^2\big)\big(v_p^2 - k\big)} \tag{7}$$

Inserting Eq. (7) into Eq. (5), we get the final expression

$$\rho(k|v_a, v_p) = \frac{2}{\pi} \frac{v_p + v_a}{(k + v_a v_p)^2} \sqrt{\frac{v_p^3 v_a^3}{\big(k - v_a^2\big)\big(v_p^2 - k\big)}} \tag{8}$$

This is plotted in Fig. 1 for an Earth orbit.

Radial Velocity Distribution

Next we determine the analytical expression for the probability density function associated with the variable v_ρ, that is, the radial velocity along an orbit. The time is considered as random and uniformly distributed within an orbital period. We start by writing

$$\rho(v_\rho|\boldsymbol{\alpha}, t) = \delta_v[v_\rho - \hat{v}_\rho(\boldsymbol{\alpha}, t)]$$

where $\hat{v}_\rho(\boldsymbol{\alpha}, t) = (\mu e/h)\sin[\theta(t) - \omega]$. Because time is random, we disintegrate it from the process:

$$\rho(v_\rho|\boldsymbol{\alpha}) = \frac{1}{T}\int_0^T \delta_v[v_\rho - \hat{v}_\rho(\boldsymbol{\alpha}, t)]\,\mathrm{d}t$$

When Eq. (1) is used,

$$\rho(v_\rho|\boldsymbol{\alpha}) = \frac{1}{T}\int_0^T \sum_i \frac{\delta_t[t - t_i]}{|\mathrm{d}\hat{v}_\rho/\mathrm{d}t|_{t=t_i}}\,\mathrm{d}t$$

where t_i are the solutions in t to the equation $\hat{v}_\rho(\boldsymbol{\alpha}, t) = v_\rho$ given by the expression

$$\sin[\theta(t_i) - \omega] = v_\rho/v_{\rho M}$$

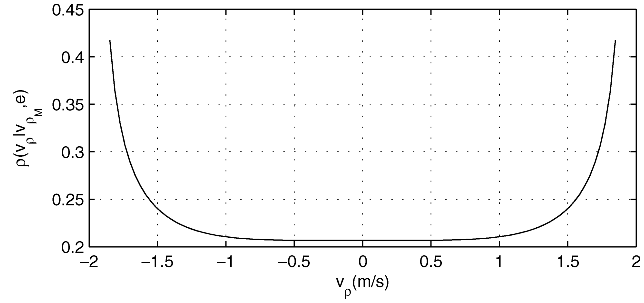

Fig. 2 Probability density function associated with radial velocity on an Earth orbit having $e = 0.4$ and $a = 20{,}000$ km.

where we introduced the maximum value of the radial velocity $v_{\rho M} = \mu e/h$. We also have

$$\left|\frac{\mathrm{d}\hat{v}_\rho}{\mathrm{d}t}\right|_{t=t_i} = v_{\rho M}\cos[\theta(t_i) - \omega]\dot{\theta}$$

$$= v_{\rho M}\cos[\theta(t_i) - \omega] b \sqrt{\frac{\mu}{a}}\frac{1}{r^2}$$

$$= \frac{2\pi}{T}\frac{\sqrt{v_{\rho M}^2 - v_\rho^2}}{\sqrt{(1-e^2)^3}}\left(1 \pm \frac{e}{v_{\rho M}}\sqrt{v_{\rho M}^2 - v_\rho^2}\right)^2 \tag{9}$$

Note that the values of the derivative are different for the two different time instants t_i contained in one orbital period. After some more algebraic manipulations, we get

$$\rho(v_\rho|v_{\rho M}, e) = \frac{1}{\pi}\frac{v_{\rho M}^2\sqrt{(1-e^2)^3}}{\sqrt{v_{\rho M}^2 - v_\rho^2}}\frac{v_{\rho M}^2 + e^2\left(v_{\rho M}^2 - v_\rho^2\right)}{\left[v_{\rho M}^2 - e^2\left(v_{\rho M}^2 - v_\rho^2\right)\right]^2} \tag{10}$$

This is shown in Fig. 2 for an Earth orbit.

Radial–Tangential Velocity Distribution

In a deterministic process, the probability density function associated with the variables v_ρ and v_θ may be written as

$$\rho(v_\rho, v_\theta|\boldsymbol{\alpha}, t) = \delta[v_\rho - \hat{v}_\rho(\boldsymbol{\alpha}, t)]\delta[v_\theta - \hat{v}_\theta(\boldsymbol{\alpha}, t)]$$

where

$$\hat{v}_\rho = v_{\rho M}\sin[\theta(t) - \omega]$$

$$\hat{v}_\theta = v_{\rho M}\{1/e + \cos[\theta(t) - \omega]\}$$

Consider time as a random variable uniformly distributed in one orbital period and the eccentricity as a random variable with which we associate a probability distribution function $f(e)$. By disintegrating these two variables we get

$$\rho(v_\rho, v_\theta|v_{\rho M}) = \frac{1}{T}\int_0^1\int_0^T \rho(v_\rho, v_\theta|\boldsymbol{\alpha}, t) f(e)\,\mathrm{d}t\,\mathrm{d}e$$

We could now use Eq. (2). However, this would require the evaluation the determinant of the Jacobian matrix. By applying instead Eq. (1) twice, we save some calculations. Start by eliminating the Dirac delta in v_ρ. Taking advantage of the calculations already done in the preceding paragraph, we get

$$\rho(v_\rho, v_\theta|v_{\rho M}) = \int_0^1\left\{\frac{1}{2\pi}\frac{\sqrt{(1-e^2)^3}}{\sqrt{v_{\rho M}^2 - v_\rho^2}}\frac{f(e)\delta[v_\theta - \hat{v}_{\theta_1}]}{\left[1 + (e/v_{\rho M})\sqrt{v_{\rho M}^2 - v_\rho^2}\right]^2}\right.$$

$$\left. + \frac{1}{2\pi}\frac{\sqrt{(1-e^2)^3}}{\sqrt{v_{\rho M}^2 - v_\rho^2}}\frac{f(e)\delta\left[v_\theta - \hat{v}_{\theta_2}\right]}{\left[1 - (e/v_{\rho M})\sqrt{v_{\rho M}^2 - v_\rho^2}\right]^2}\right\}\mathrm{d}e \tag{11}$$

where $\hat{v}_{\theta_{1,2}}(v_{\rho M}, e) = v_{\rho M}/e \pm \sqrt{(v_{\rho M}^2 - v_\rho^2)}$. Now apply Eq. (1) again to eliminate the two Dirac deltas in v_θ. Taking into account

$$\left|\frac{\mathrm{d}\hat{v}_{\theta_{1,2}}}{\mathrm{d}e}\right| = \frac{v_{\rho M}}{e^2} \tag{12}$$

and that the two solutions in e to the two equations $v_\theta = \hat{v}_{\theta_{1,2}}$ are

$$e_{1,2}^* = v_\theta \Big/ \left(v_{\rho M} \mp \sqrt{v_{\rho M}^2 - v_\rho^2}\right)$$

we get a final expression of the form

$$\rho(v_\rho, v_\theta|v_{\rho M}) = \frac{1}{2\pi}\frac{v_{\rho M}}{v_\theta^2\sqrt{v_{\rho M}^2 - v_\rho^2}}\sum_{i=1}^{2}\sqrt{\left(1 - e_i^{*2}\right)^3} f\left(e_i^*\right) \tag{13}$$

This expression still contains $v_{\rho M}$ as a conditioning variable. This variable might also be disintegrated leading to a formula solvable by quadrature.

Last Example

We derive a formula for the probability distribution function associated with the variables r, v_ρ, and v_θ,

$$\rho(r, v_\rho, v_\theta|\boldsymbol{\alpha}, t) = \delta[r - \hat{r}(\boldsymbol{\alpha}, t)]\delta[v_\rho - \hat{v}_\rho(\boldsymbol{\alpha}, t)]\delta[v_\theta - \hat{v}_\theta(\boldsymbol{\alpha}, t)]$$

First consider the time as random and uniformly distributed in one period, and then the eccentricity as random and described by a probability density function f. We get the expression

$$\rho(r, v_\rho, v_\theta|v_{\rho M}) = \left(\frac{1}{2\pi}\frac{v_{\rho M}}{v_\theta^2\sqrt{v_{\rho M}^2 - v_\rho^2}}\right)$$

$$\times\left\{\sum_{i=1}^{2}\sqrt{\left(1 - e_i^{*2}\right)^3} f\left(e_i^*\right)\delta[r - \hat{r}_i(v_\theta, v_{\rho M})]\right\} \tag{14}$$

where $\hat{r}_i(v_\theta, v_{\rho M}) = (\mu e_i^*/v_{\rho M})1/v_\theta$. Next we consider $v_{\rho M}$ as random and with a probability distribution function g and we eliminate the last Dirac delta by disintegrating the random variable from the orbital process. Using

$$\left|\frac{\mathrm{d}\hat{r}_i}{\mathrm{d}v_{\rho M}}\right| = \frac{\mu}{v_{\rho M}^2\sqrt{v_{\rho M}^2 - v_\rho^2}}$$

and that the only solution in $v_{\rho M}$ of the equation $r = \hat{r}_i$ is

$$v_{\rho M}^* = \mu \Big/ \sqrt{2\mu r - r^2 v_\rho^2}$$

we apply the Dirac delta property again stated in Eq. (1) and after some basic algebraic manipulations find a final expression in the form

$$\rho(r, v_\rho, v_\theta) = \frac{\mu^2}{2\pi}\frac{1}{v_\theta^2}\frac{g\left(v_{\rho M}^*\right)}{\sqrt{\left(2\mu r - r^2 v_\rho^2\right)^3}}\sum_{i=1}^{2}\sqrt{\left(1 - e_i^{*2}\right)^3} f\left(e_i^*\right) \tag{15}$$

This probability density function is no longer conditioned by the knowledge of any initial condition. This expression describes the velocity distribution of a satellite, or of a group of objects, whose uncertain orbits are described via the density functions associated with two orbital parameters, the eccentricity e and the maximum radial velocity ρ_M. We note that the other orbital parameters may also be considered as random, but they do not influence this joint distribution that remains valid. Equation (15) is valid under the hypothesis that e and ρ_M are uncorrelated random variables; a similar expression may also be obtained if a joint probability distribution function is introduced instead.

Conclusions

We use a formalism based on the disintegration (randomization) of the orbital parameters from an initial density function written in terms of Dirac's delta to find analytical expressions for the distributions of Keplerian velocities. The method used represents an alternative formalism to the standard change of variable and allows for an easy transition from astrodynamic formulas to their statistical counterparts. Whenever the radial velocity distribution is required (alone or in a joint distribution), the use of the maximum radial velocity as an orbital parameter allows for compact and elegant expressions. We also show how the joint distribution of radial distance, radial velocity, and tangential velocity admits an algebraic analytical expression in its unconditioned form. The statistical description of the orbital debris environment as well as the evaluation of satellite collision probabilities may improve using the results here derived. These new formulas will help our understanding of situations in which orbiting objects have to be described probabilistically.

References

[1] Izzo, D., "Statistical Modelling of Sets of Orbits," Ph.D. Dissertation, Dept. of Mathematical Methdods and Models, Univ. La Sapienza, Rome [online], URL: http://www.dmmm.uniroma1.it/dottorato/izzo.pdf [cited 30 June 2005].

[2] Izzo, D., and Valente, C., "A Mathematical Model Representing the Statistical Properties of Sets of Orbits," *Acta Astronautica*, Vol. 54, No. 8, 2004, pp. 541–546.

[3] Izzo, D., "Effects of Orbital Parameters Uncertainties," *Journal of Guidance, Control, and Dynamics*, Vol. 28, No. 2, 2005, pp. 298–305.

[4] Meshcheryakov, S., "Calculation of the Density of the Space Objects Using Dirac Functions," *Theoretical and Experimental Investigations on the Problems of General Physics*, edited by N.A. Animof, TSNIIMASH, Moscow, 2001, pp. 59–62.

[5] Meshcheryakov, S., "Use of Generalized Functions for the Definition of Collision Integrals in Orbital Motion," *Proceedings of the Fourth European Conference on Space Debris*, ESA SP-587, Noordwijk, The Netherlands, 2005.

[6] Au, C., and Tam, J., "Transforming Variables Using the Dirac Generalized Function," *American Statistician*, Vol. 53, No. 3, 1999, pp. 270–272.

[7] Kessler, D., Zhang, J., Matney, M., Eichler, P., Reynolds, R., Anz-Meador, P., and Stansbery, E., "A Computer-Based Orbital Debris Environment Model for Spacecraft Design and Observation in Low Earth Orbit," NASA TM-104825, Nov. 1996.

[8] Dennis, N., "Probabilistic Theory and Statistical Distribution of Earth Satellites," *Journal of the British Interplanetary Society*, Vol. 25, 1989, pp. 333–376.

[9] Nazarenko, "The Devolpment of a Statistical Theory of a Satellite Ensemble Motion and its Application to Space Debris Modelling," *Proceedings of the Second European Conference on Space Debris*, ESA SP-393, Noordwijk, The Netherlands, 1997.

[10] Kessler, D., "Derivation of the Collision Probability Between Orbiting Objects: The Lifetime of the Jupiter's Outer Moons," *Icarus*, Vol. 48, No. 1, 1981, pp. 39–48.

[11] Tuzzolino, A., Economou, T., McKibben, R., Simpson, J., BenZvi, S., Blackburn, L., Voss, H., and Gursky, H., "Final Results from the Space Dust (SPADUS) Instrument Flown Aboard the Earth-Orbiting ARGOS Spacecraft," *Planetary and Space Science*, No. 53, No. 9, 2005, pp. 902–923.

[12] Papoulis, A., *Probability Random Variables and Stochastic Processes*, McGraw–Hill, New York, 1984.